

Fluctuation-dissipation theorem for thermo-refractive noise.

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We introduce a simple prescription for calculating the spectra of thermal fluctuations of temperature-dependent quantities of the form $\delta\hat{T}(t) = \int d^3\vec{r} \delta T(\vec{r}, t) q(\vec{r})$. Here $T(\vec{r}, t)$ is the local temperature at location \vec{r} and time t , and $q(\vec{r})$ is an arbitrary function. As an example of a possible application, we compute the spectrum of thermo-refractive coating noise in LIGO, and find a complete agreement with the previous calculation of Braginsky, Gorodetsky and Vyatchanin. Our method has computational advantage, especially for non-regular or non-symmetric geometries, and for the cases where $q(\vec{r})$ is non-negligible in a significant fraction of the total volume.

I. INTRODUCTION AND MAIN RESULTS

The theory of time-dependent thermodynamical fluctuations has been extensively developed for the past century. One of the fundamental results in this field is the Fluctuation-dissipation Theorem (FdT), which was originally formulated by Callen and Welton in 1951 [1]. Several different formulations of the FdT have been introduced since then, see [2] for a review. In this short paper we develop a formulation which is suitable for calculating the spectra $S_{\delta\hat{T}}(\omega)$ of thermal fluctuations of temperature-dependent quantities of the form

$$\delta\hat{T} = \int d^3\vec{r} \delta T(\vec{r}, t) q(\vec{r}). \quad (1)$$

Here $\delta T(\vec{r}, t)$ is the local temperature at location \vec{r} and time t , and $q(\vec{r})$ is an arbitrary function.

Computations of this kind may be relevant for the design of interferometric gravitational-wave detectors like the Laser Interferometer Gravitational Wave Observatory (LIGO). Random thermal fluctuations are expected to be the dominant noise source for LIGO at frequencies between 10 and 100 Hz; see, e.g., [3]. This noise is mostly due to the thermal motion of the LIGO mirror surfaces, and has been extensively studied both theoretically and experimentally by several groups around the world; see, e.g., [4] and references therein. More recently Braginsky, Gorodetsky, and Vyatchanin ([5], hereafter BGV) have identified a different kind of thermal noise, which they called the thermo-refractive (TR) noise. The TR noise can be understood as follows:

Consider a laser beam which passes through (the part of) one of LIGO test masses. The index of refraction of the test-mass material is strongly temperature-dependent. Thus local thermodynamical fluctuation in the temperature $\delta T(\vec{r})$ result in the fluctuation of the overall phase of the laser beam, as measured by an interferometric experiment. For small expected temperature fluctuations, the variation of the phase is of the same form as in Eq. (1). Thus a computation of the thermo-refractive noise amounts to that of the fluctuations in $\delta\hat{T}$. BGV have computed the TR noise for a plane-parallel geometry of the thin mirror coating. Their computation

was based on the multi-dimensional Langevin equations and involved a calculation of the spacial temperature correlation functions. While BGV approach works fine for simple geometries, it becomes numerically cumbersome in more general geometries, i.e. in a case when the light goes through one of the test masses and the light-beam radius is comparable to that of the test mass.

By contrast, our computational approach will be based on a direct application of the FdT, in the spirit of our earlier treatment of the mechanical thermal noise ([6], hereafter L98). We will show that in order to compute the fluctuations in \hat{T} , one needs to perform the following mental experiment consisting of 3 steps:

1. Periodically inject entropy into the medium, with the volume density of the entropy injection given by

$$\frac{\delta s(\vec{r})}{dV} = F_0 \cos(\omega t) q(\vec{r}), \quad (2)$$

where F_0 is an arbitrarily small constant.

2. Track all thermal relaxation processes in the system (e.g., the heat exchange between different parts of the system) which occur as a result of the periodic entropy injection. Calculate the total entropy production rate and hence the total dissipated power W_{diss} which occurs as a result of the thermal relaxation.

3. Evaluate the spectral density of fluctuations in $\delta\hat{T}$ via the formula below:

$$S_{\delta\hat{T}}(f) = \frac{8k_B T}{\omega^2} \frac{W_{\text{diss}}}{F_0^2}; \quad (3)$$

cf. Eq. (1) of L98. Here $\omega = 2\pi f$.

In the following section we give a proof of the above prescription. In section 3, as an illustration of the method, we compute the thermo-refractive coating noise in the BGV geometry. We conclude with some brief general discussion in section 4.

II. PROOF

It may well be possible to verify Eq. (3) by appealing to one of the existing formulations of the FdT, which

treat the fluctuations of generalized thermodynamic variable (see [2]). However, the gravitational-wave community (including this author) is much more familiar with the FdT for a generalized *mechanical* coordinate of the system, as given by Callen and Welton in 1951 [1]. We thus think it is instructive to construct a proof of Eq. (3) using a mechanical coordinate.

For this, we mentally introduce a set of non-intrusive mechanical thermometers into the system. Our thermometers are localized ensembles of identical harmonic oscillators, which are assumed to

- (a) be sufficiently densely packed into the system (we will make this more precise shortly),
- (b) have total mass and heat capacity which are vanishingly small compared with those of the original system (for the latter it is sufficient to assume that the number of thermometers is much smaller than the number of particles in the system),
- (c) have a proper angular frequency ω_0 which is much higher than the angular frequencies $2\pi f$ at which $S_{\hat{T}}(f)$ is computed, and
- (d) be thermally coupled to the system on a timescale much shorter than $1/f$.

Consider now an operator

$$\hat{T}_1 = \sum_i x_i^2, \quad (4)$$

where x_i is the displacement of the i 'th oscillator. For densely packed oscillators, the equation above can be written as

$$\hat{T}_1 = \int d^3r n(\vec{r}) < x^2(\vec{r}) >, \quad (5)$$

where $n(\vec{r})$ is the spacial density of the oscillators, and $< x^2(\vec{r}) >$ is the average x^2 at a radius \vec{r} . Note that as the number of oscillators increases, the Eq. (5) becomes better defined and more precise. For sufficiently dense packing,

$$< x^2(\vec{r}) > = \frac{k_B}{m\omega_0^2} T(\vec{r}), \quad (6)$$

where k_B is the Boltzmann constant, and m is the oscillator mass. Here, when we say “sufficiently dense”, we mean that in a minimum volume for which the notion of local temperature $T(\vec{r})$ is meaningful, there should be a number of oscillators $\gg 1$. If such volume has $N_1 \gg 1$ particles, then it should have N_2 oscillators where $N_1 \gg N_2 \gg 1$.

We now choose the density of the oscillators be

$$n(\vec{r}) = \frac{m\omega_0^2}{k_B} q(\vec{r}), \quad (7)$$

where $q(\vec{r})$ is the form factor from Eq. (1); we can always rescale $q(\vec{r})$ so that the oscillators are sufficiently densely packed. With this choice of $n(\vec{r})$, the dynamical variable \hat{T}_1 closely tracks the thermodynamical variable \hat{T} .

We can now use the Callen-Welton formulation of the FdT to find the fluctuation $S_{\hat{T}}(f)$ in \hat{T}_1 . We follow

closely the steps described in L98:

Step 1. We introduce a periodic perturbation of the form of the interaction Hamiltonian

$$H_{\text{int}} = -F_0 \cos(2\pi f t) \hat{T}_1, \quad (8)$$

and consider response of the system to this perturbation. From Eq. (4) we see that physically such perturbation amounts to a periodic change in the rigidity of each of the oscillator, or, equivalently, in a periodic change $\delta\omega_0$ in the oscillator proper frequency:

$$\frac{\delta\omega_0}{\omega_0} = \frac{F_0 \cos(2\pi f t)}{m\omega_0^2}. \quad (9)$$

If the oscillators were not thermally coupled to the system, their energy would track adiabatically the change in the proper frequency, $\delta E/E = \delta\omega_0/\omega_0$. Once the thermal coupling is included, their energy change is given by

$$\frac{\delta E_i}{E_i} = \frac{\delta\omega_0}{\omega_0} - \frac{\delta Q_i}{E_i}, \quad (10)$$

where δQ_i is the energy input from the i 'th oscillator to the local thermal bath. By construction the oscillators' thermal coupling to the system is much more rapid than $1/f$, and thus on average their energy does not change (it remains $k_B T$). Therefore, once averaged over the local volume dV , Eq. (10) gives

$$\delta Q = \frac{F_0 \cos(2\pi f t)}{m\omega_0^2} \times k_B T n(\vec{r}) \times dV. \quad (11)$$

We now substitute Eq. (7) into the above equation, and get

$$\frac{\delta s}{dV} = \frac{1}{T} \frac{\delta Q}{dV} = F_0 \cos(2\pi f t) q(\vec{r}). \quad (12)$$

This is the density of the local entropy injection as a result of the periodic perturbation driven by H_{int} .

Step 2. We compute the total power W_{diss} which is dissipated in the system as a result of the periodic forcing. This means that we compute the total entropy production in the system; from Eq. (12) we see that it is zero to first order in F_0 . Not so to the second order: the entropy injection leads to temperature inhomogeneities which in turn lead to thermal relaxation processes, i.e. to the heat exchange between different parts of the system. The thermal relaxation leads to the net entropy production rate which is second order in F_0 .

Step 3. We compute $S_{\hat{T}}(f)$ by using Eq. (3). This completes justification for and physical description of the computational procedure outlined in the introduction. In next section we work out a practical example, as a useful crosscheck of our approach.

III. THERMO-REFRACTIVE NOISE IN LIGO MIRROR COATING

BGV studied the thermo-refractive noise in the thin optical coating of LIGO mirrors. We re-derive their result using the direct approach developed above.

As BGV have explained, the relevant variable for the coating thermo-refractive fluctuation is

$$\hat{\delta T} = \frac{1}{\pi r_0^2 l} \int_{-\infty}^{\infty} dx dy \int_0^{\infty} dz \delta T(\vec{r}, t) e^{-(x^2+y^2)/r_0^2} e^{-z/l}, \quad (13)$$

where r_0 is the effective beam size, l is the effective coating thickness, x and y are the rectangular coordinates along the mirror face, and z is the coordinate along the beam, chosen so that $z = 0$ at the coating outside boundary. Within our approach, we need to, as a thought experiment, inject an oscillating entropy perturbation of the form in Eq. (2), where

$$q(\vec{r}) = \frac{1}{\pi r_0^2 l} e^{-(x^2+y^2)/r_0^2} e^{-z/l}. \quad (14)$$

The heat injection leads to periodic temperature perturbation $\delta T(\vec{r})$, which we compute below. We can then find the dissipated power [see Eq. (35.1) of Landau and Lifshitz [7] and Eq. (5) of [8]]:

$$W_{\text{diss}} = \int d^3 r \frac{\kappa}{T} \langle (\nabla \delta T)^2 \rangle, \quad (15)$$

where κ is the thermal conductivity, and $\langle \dots \rangle$ stands for averaging over the oscillation period $2\pi/\omega$.

As BGV have noted, the thermal diffusion lengthscale $l_{\text{th}} = \sqrt{\kappa/(C\rho\omega)}$ satisfies the following inequality:

$$r_0 \gg l_{\text{th}} \gg l. \quad (16)$$

Therefore, (1) the heat diffusion is almost exclusively in the z direction, and (2) we can consider all of the entropy to be injected at the outer coating boundary, with the surface density

$$\frac{\delta s}{dA} = \frac{F_0 \cos(\omega t)}{\pi r_0^2} e^{-(x^2+y^2)/r_0^2}. \quad (17)$$

The temperature perturbation satisfies the diffusion equation

$$\frac{\partial \delta T}{\partial t} = D \frac{\partial^2 \delta T}{\partial z^2}, \quad (18)$$

and the boundary condition

$$-\kappa \left(\frac{\partial \delta T}{\partial z} \right)_{z=0} = T \frac{\partial}{\partial t} \frac{\delta s}{dA}. \quad (19)$$

The latter simply states that all the heat injected at the boundary is transported inwards[9]. Here $D = \kappa/(\rho C)$, where ρ is the density and C is the heat capacity.

It is now straightforward to find δT and its gradient:

$$\frac{\partial T}{\partial z} = \frac{T F_0 \omega}{\kappa \pi r_0^2} \exp \left(-\sqrt{\frac{\omega}{2D}} z \right) \sin \left(\sqrt{\frac{\omega}{2D}} z - \omega t \right) e^{-(x^2+y^2)/r_0^2}. \quad (20)$$

We can now find the dissipated power:

$$W_{\text{diss}} \simeq \int d^3 r \frac{\kappa}{T} \left\langle \left(\frac{\partial \delta T}{\partial z} \right)^2 \right\rangle = \frac{F_0^2 \omega^2 T}{4 \pi r_0^2 \kappa} \sqrt{\frac{D}{2\omega}}. \quad (21)$$

Finally, from Eq. (3), we get

$$S_{\delta T}(f) = \frac{\sqrt{2} k_B T^2}{\pi r_0^2 \sqrt{\omega C \rho \kappa}}. \quad (22)$$

This result is identical to Eq. (9) of BGV.

IV. DISCUSSION

Our direct approach to the thermo-refractive noise has 2 advantages, as compared to the previously used methods:

1. It is computationally easier, especially when the form-factor $q(\vec{r})$ is significant over a large fraction of the system's volume. For example, a broad beam of light passing through a LIGO test mass will be sensitive to the thermo-refractive fluctuations in a large fraction of the test mass' volume.

2. It elucidates geometries and material properties which may make the thermo-refractive noise high. The thermal relaxation which is induced in the process of our mental experiment depends on the spacial variation of the form-factor $q(\vec{r})$ and (less obviously) on that of the heat capacity.

Future work may include unified simultaneous modelling of thermo-refractive and thermoelastic noises, since they both originate from T -fluctuations and may be highly correlated.

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